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Properties of certain new special polynomials associated with Sheffer sequences

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Abstract

In this article, the Laguerre-Gould Hopper polynomials are combined with Sheffer sequences to introduce certain mixed type special polynomials. Certain important properties of these polynomials are established. Further, operational and integral representations for these mixed polynomials are derived.

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1. Introduction and preliminaries

We recall that the polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ ($s_n(x)$ being a polynomial of degree n) is called Sheffer A-type zero [39, p.222 (Theorem 72)], (which we shall hereafter call Sheffer-type), if $s_n(x)$ possesses the exponential generating function of the form

$$A(t)\exp\left(xH(t)\right) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where $A(t)$ and $H(t)$ have (at least the formal) expansions:

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \qquad A_0 \neq 0 \tag{1.2a}$$

and

$$H(t) = \sum_{n=1}^{\infty} H_n \frac{t^n}{n!}, \qquad H_1 \neq 0, \tag{1.2b}$$

respectively.

Also, in view of the following result [40, p.17], the Sheffer sequences can be alternatively defined as:

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Let $f(t)$ be a delta series and let $g(t)$ be an invertible series of the following form:

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \qquad f_0 = 0,\ f_1 \neq 0 \tag{1.3a}$$

and

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \qquad g_0 \neq 0. \tag{1.3b}$$

Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions

$$\langle g(t)\ f(t)^k |\ s_n(x) \rangle = n!\ \delta_{n,k}\ , \qquad \forall\ n,k \geq 0. \tag{1.4}$$

According to Roman [40, p.18 (Theorem 2.3.4)], the polynomial sequence $s_n(x)$ is uniquely determined by two (formal) power series given by equations (1.3a) and (1.3b). The exponential generating function of $s_n(x)$ is then given by

$$\frac{1}{g\big(f^{-1}(t)\big)} \exp\big(xf^{-1}(t)\big) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{1.5}$$

for all x in $\mathbb{C}$, where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

In view of equations (1.1) and (1.5), we have

$$A(t) = \frac{1}{g\big(f^{-1}(t)\big)} \tag{1.6a}$$

and

$$H(t) = f^{-1}(t). \tag{1.6b}$$

The sequence $s_n(x)$ in equation (1.4) is the Sheffer sequence for the pair $\big(g(t), f(t)\big)$. The Sheffer sequence for $\big(1, f(t)\big)$ is called the associated Sheffer sequence for $f(t)$ and the Sheffer sequence for $\big(g(t), t\big)$ becomes the Appell sequence for $g(t)$ [40, p. 17]. Properties of Appell and Sheffer sequences are naturally handled within the framework of modern classical umbral calculus by Roman [40].

The idea of monomiality arised within the context of poweroid, suggested by J. F. Steffenson [41]. The monomiality principle has been proved to be a powerful tool for the investigation of the properties of a wide class of polynomials like Sheffer polynomials, see for example [15]. The monomiality principle is reformulated and developed by Dattoli [9], according to which, the polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is "quasi-monomial", provided there exist two operators $\hat{M}$ and $\hat{P}$ playing, respectively, the role of multiplicative and derivative operators, for the family of polynomials. These operators satisfy the following identities:

$$\hat{M}\,\{p_n(x)\} = p_{n+1}(x) \tag{1.7}$$

and

$$\hat{P}\,\{p_n(x)\} = n\ p_{n-1}(x), \tag{1.8}$$

for all $n \in \mathbb{N}$. The operators $\hat{M}$ and $\hat{P}$ also satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1} \tag{1.9}$$

and thus display the Weyl group structure. If the considered polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can easily be derived from those of the $\hat{M}$ and $\hat{P}$ operators. In fact:

(i) Combining recurrences (1.7) and (1.8), we have

$$\hat{M}\,\hat{P}\{p_n(x)\} = n\;p_n(x), \tag{1.10}$$

which can be interpreted as the differential equation satisfied by $p_n(x)$, if $\hat{M}$ and $\hat{P}$ have a differential realization.

(ii) Assuming here and in the sequel $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as:

$$p_n(x) = \hat{M}^n\{1\}, \tag{1.11}$$

which yields the series definition for $p_n(x)$.

(iii) Identity (1.11) implies that the exponential generating function of $p_n(x)$ can be given in the form:

$$\exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \qquad |t| < \infty. \tag{1.12}$$

We recall that the Laguerre-Gould Hopper polynomials (LGHP) ${}_LH_n^{(m,r)}(x, y, z)$ are introduced in [29] and defined by the following generating function:

$$C_0(-xt^m)\exp(yt + zt^r) = \sum_{n=0}^{\infty} {}_LH_n^{(m,r)}(x, y, z)\frac{t^n}{n!}, \tag{1.13}$$

where $C_0(x)$ denotes the Bessel-Tricomi function of order 0. The nth order Bessel-Tricomi function $C_n(x)$ defined by the following series [9, p.150]:

$$C_n(x) \;=\; x^{-\frac{n}{2}}J_n(2\sqrt{x}) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k\;x^k}{k!\;(n+k)!}, \qquad n = 0, 1, 2, \ldots\;, \tag{1.14}$$

with $J_n(x)$ being the ordinary cylindrical Bessel function of first kind [1]. The 0^{th}-order Bessel-Tricomi function $C_0(x)$ is also given by the following operational definition:

$$C_0(\alpha x) = \exp\left(-\alpha D_x^{-1}\right)\{1\}, \tag{1.15}$$

where D_x^{-1} denotes the inverse derivative operator and

$$D_x^{-n}\{1\} = \frac{x^n}{n!}. \tag{1.16}$$

The LGHP ${}_LH_n^{(m,r)}(x,y,z)$ are shown to be quasi-monomial under the action of the following multiplicative and derivative operators [29]:

$$\hat{M}_{LH} := y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}} \tag{1.17}$$

and

$$\hat{P}_{LH} := \frac{\partial}{\partial y}, \tag{1.18}$$

respectively.

The LGHP ${}_LH_n^{(m,r)}(x,y,z)$ are important as they contain several special polynomials as their particular cases. For suitable values of the indices and variables the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to these special polynomials. We mention these special cases in the following table:

Table 1. Special cases of the LGHP ${}_LH_n^{(m,r)}(x,y,z)$

S. No.	Values of the Indices and Variables	Relation Between the LGHP ${}_LH_n^{(m,s)}(x,y,z)$ and its Special Case	Name of the Known Polynomials	Generating Functions of the Known Polynomials
I.	$m=1,\ r=2;$ $x \to -x$	${}_LH_n^{(1,2)}(-x,y,z) = {}_LH_n(x,y,z)$	3-Variable Laguerre - Hermite [23]	$C_0(xt)\exp\left(yt+zt^2\right) = \sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!}$
II.	$m=1,\ r=2;$ $z=-\frac{1}{2},$ $x \to -x$	${}_LH_n^{(1,2)}(-x,y,-\frac{1}{2}) = {}_LH_n^{*}(x,y)$	2-Variable Laguerre - Hermite [24]	$C_0(xt)\exp\left(yt-\frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} {}_LH_n^{*}(x,y)\frac{t^n}{n!}$
III.	$m=1,\ r=2;$ $y=1,\ z\to y,$ $x\to -x$	${}_LH_n^{(1,2)}(-x,1,y) = \varphi_n(x,y)$	Laguerre - Hermite type [22]	$C_0(xt)\exp\left(t+yt^2\right) = \sum_{n=0}^{\infty} \varphi_n(x,y)\frac{t^n}{n!}$
IV.	$x=0$	${}_LH_n^{(m,r)}(0,y,z) = H_n^{(r)}(y,z)$	Gould-Hopper [27]	$\exp\left(yt+zt^r\right) = \sum_{n=0}^{\infty} H_n^{(r)}(y,z)\frac{t^n}{n!}$
V.	$z=0$	${}_LH_n^{(m,r)}(x,y,0) = {}_mL_n(x,y)$	2-Variable generalized Laguerre [13]	$C_0\left(-xt^m\right)\exp\left(yt\right) = \sum_{n=0}^{\infty} {}_mL_n(x,y)\frac{t^n}{n!}$
VI.	$r=m;\ x=0,$ $y\to -D_x^{-1},$ $z\to y$	${}_LH_n^{(m,m)}(0,-D_x^{-1},y)$ $= {}_{[m]}L_n(x,y)$	2-Variable generalized Laguerre type [11]	$C_0(xt)\exp\left(yt^m\right) = \sum_{n=0}^{\infty} {}_{[m]}L_n(x,y)\frac{t^n}{n!}$
VII.	$r=m-1;\ x=0,$ $y\to x,\ z\to y$	${}_LH_n^{(m,m-1)}(0,x,y) = U_n^{(m)}(x,y)$	Generalized Chebyshev [10]	$\exp\left(xt+yt^{m-1}\right) = \sum_{n=0}^{\infty} U_n^{(m)}(x,y)\frac{t^n}{n!}$
VIII.	$m=1;\ z=0,$ $x\to -x$	${}_LH_n^{(1,r)}(-x,y,0) = L_n(x,y)$	2-Variable Laguerre [20]	$C_0(xt)\exp(yt) = \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}$
IX.	$m=1;\ z=0,$ $x\to y,$ $y\to -D_x^{-1}$	$n!\,{}_LH_n^{(1,r)}(y,-D_x^{-1},0)$ $= R_n(x,y)$	2-Variable Legendre [17]	$C_0(xt)C_0(-yt) = \sum_{n=0}^{\infty} \frac{R_n(x,y)}{n!}\frac{t^n}{n!}$
X.	$x=0,\ y\to x,$ $z\to yD_yy$	${}_LH_n^{(m,r)}(0,x,yD_yy) = e_n^{(r)}(x,y)$	2-Variable truncated of order s [11] (see [14])	$\frac{1}{(1-yt^r)}\exp(xt) = \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!}$
XI.	$r=2;\ x=0$	${}_LH_n^{(m,2)}(0,y,z) = H_n(y,z)$	2-Variable Hermite-Kampé de Fériet [3]	$\exp\left(yt+zt^2\right) = \sum_{n=0}^{\infty} H_n(y,z)\frac{t^n}{n!}$
XII.	$r=2;\ x=0,$ $y\to D_x^{-1},\ z\to y$	${}_LH_n^{(m,2)}(0,D_x^{-1},y)$ $= G_n(x,y)$	Hermite type [12]	$C_0(-xt)\exp\left(yt^2\right) = \sum_{n=0}^{\infty} G_n(x,y)\frac{t^n}{n!}$
XIII.	$m=2;\ z=0,$ $x\to(\frac{x^2-1}{4}),$ $y\to x$	${}_LH_n^{(2,r)}\left(\frac{x^2-1}{4},x,0\right) = P_n(x)$	Legendre [39]	$C_0\left(-\frac{(x^2-1)}{4}t^2\right)\exp(xt) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}$
XIV.	$x\to yD_yy,$ $y\to x$	${}_LH_n^{(m,r)}(yD_yy,x,z)$ $= H_n^{(r,m)}(x,y,z)$	3-Variable generalized Hermite [18]	$\exp\left(xt+yt^m+zt^r\right) = \sum_{n=0}^{\infty} H_n^{(r,m)}(x,y,z)\frac{t^n}{n!}$
XV.	$m=2,\ r=3;$ $x\to zD_zz,$ $y\to x,\ z\to y$	${}_LH_n^{(2,3)}(zD_zz,x,y)$ $= H_n^{(3,2)}(x,y,z)$	Bell-type [21]	$\exp\left(xt+yt^3+zt^2\right) = \sum_{n=0}^{\infty} H_n^{(3,2)}(x,y,z)\frac{t^n}{n!}$

The correspondence given in Table 1 can be used to derive the results for the polynomials related to the LGHP ${}_LH_n^{(m,r)}(x,y,z)$.

In this paper, the families of Laguerre-Gould Hopper based Sheffer polynomials are introduced by using the concepts and the methods associated with monomiality principle. In Section 2, we introduce the Laguerre-Gould Hopper based Sheffer polynomials (LGHSP) ${}_{L}{}_{H^{(m,r)}}s_n(x,y,z)$ and frame these polynomials within the context of monomiality principle formalism. In Section 3, we consider some examples of these polynomials. In Section 4, the operational and integral representations for the Laguerre-Gould Hopper based Sheffer polynomials are established. In Section 5, results are obtained for the members of Laguerre-Gould Hopper based Sheffer and Laguerre-Gould Hopper based associated Sheffer polynomial families by considering some members of the Sheffer and associated Sheffer families respectively.

2. Laguerre-Gould Hopper based Sheffer polynomials

To introduce the Laguerre-Gould Hopper based Sheffer polynomials (LGHSP) denoted by ${}_{L}{}_{H^{(m,r)}}s_n(x,y,z)$, we prove the following result:

Theorem 2.1. *The Laguerre-Gould Hopper based Sheffer polynomials* ${}_{L}{}_{H^{(m,r)}}s_n(x,y,z)$ *are defined by the generating function*

$$\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)=\sum_{n=0}^{\infty}{}_{L}{}_{H^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!},\tag{2.1}$$

or, equivalently

$$A(t)C_0\left(-x\big(H(t)\big)^m\right)\exp\left(yH(t)+z\big(H(t)\big)^r\right)=\sum_{n=0}^{\infty}{}_{L}{}_{H^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!}.\tag{2.2}$$

Proof. Replacing x in the l.h.s. and r.h.s of equation (1.5) by the multiplicative operator $\hat{M}_{LH}$ of the LGHP ${}_LH_n^{(m,r)}(x,y,z)$, we have

$$\frac{1}{g\big(f^{-1}(t)\big)}\exp\left(\hat{M}_{LH}f^{-1}(t)\right)=\sum_{n=0}^{\infty}s_n\big(\hat{M}_{LH}\big)\frac{t^n}{n!}.\tag{2.3}$$

Using the expression of $\hat{M}_{LH}$ given in equation (1.17) and then decoupling the exponential operator in the l.h.s. of the resultant equation by using the Crofton-type identity [16, p. 12]

$$f\left(y+m\lambda\frac{d^{m-1}}{dy^{m-1}}\right)\{1\}=\exp\left(\lambda\frac{d^m}{dy^m}\right)\Big\{f(y)\Big\},\tag{2.4}$$

we find

$$\frac{1}{g\big(f^{-1}(t)\big)}\exp\left(z\frac{\partial^r}{\partial y^r}\right)\exp\left(\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)f^{-1}(t)\right)=\sum_{n=0}^{\infty}s_n\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)\frac{t^n}{n!},$$

which on further use of identity (2.4) gives

$$\frac{1}{g\big(f^{-1}(t)\big)}\exp\Big(z\frac{\partial^r}{\partial y^r}\Big)\exp\Big(D_x^{-1}\frac{\partial^m}{\partial y^m}\Big)\exp(yf^{-1}(t))=\sum_{n=0}^{\infty}s_n\Big(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\Big)\frac{t^n}{n!}. \tag{2.5}$$

Now, expanding the second exponential in the l.h.s. of equation (2.5) and using definition (1.15), we find

$$\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\Big(z\frac{\partial^r}{\partial y^r}\Big)\exp(yf^{-1}(t))=\sum_{n=0}^{\infty}s_n\Big(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\Big)\frac{t^n}{n!}. \tag{2.6}$$

Again, expanding the first exponential in the l.h.s. of equation (2.6) and denoting the resultant LGHSP in the r.h.s. by ${}_LH^{(m,r)}s_n(x,y,z)$, that is

$${}_LH^{(m,r)}s_n(x,y,z)=s_n(\hat{M}_{LH})=s_n\Big(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\Big), \tag{2.7}$$

we get assertion (2.1). Also, in view of equations (1.6a) and (1.6b), generating function (2.1) can be expressed equivalently as equation (2.2).

In order to show that the LGHSP ${}_LH^{(m,r)}s_n(x,y,z)$ satisfy the monomiality property, we prove the following result:

Theorem 2.2. *The Laguerre-Gould Hopper based Sheffer polynomials* ${}_LH^{(m,r)}s_n(x,y,z)$ *are quasi-monomial under the action of the following multiplicative and derivative operators:*

$$\hat{M}_{LHs}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}-\frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}, \tag{2.8a}$$

or, equivalently

$$\hat{M}_{LHs}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)H'\left(H^{-1}(\partial_y)\right)+\frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)} \tag{2.8b}$$

and

$$\hat{P}_{LHs}=f(\partial_y), \tag{2.9a}$$

or, equivalently

$$\hat{P}_{LHs}=H^{-1}(\partial_y), \tag{2.9b}$$

respectively, where $\partial_y:=\frac{\partial}{\partial y}$.

Proof. Consider the following identity:

$$\partial_y\ \left\{\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)\right\}=f^{-1}(t)\ \exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right). \tag{2.10}$$

Since f^{-1} denotes the compositional inverse of the function f and $f(t)$ has an expansion (1.3a) in powers of t, therefore we have

$$f(\partial_y)\ \left\{\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)\right\}=t\ \exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right). \tag{2.11}$$

Differentiating equation (2.3) partially with respect to t and in view of relation (2.7), we find

$$\left(\left(\hat{M}_{LH}-\frac{g'\left(f^{-1}(t)\right)}{g\left(f^{-1}(t)\right)}\right)\frac{1}{f'(f^{-1}(t))}\right)\frac{1}{g\big(f^{-1}(t)\big)}\exp\left(\hat{M}_{LH}f^{-1}(t)\right)=\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_{n+1}(x,y,z)\frac{t^n}{n!},$$

which on using monomiality principle equation (1.12) with $t=f^{-1}(t)$ gives

$$\left(\left(\hat{M}_{LH}-\frac{g'\left(f^{-1}(t)\right)}{g(f^{-1}(t))}\right)\frac{1}{f'(f^{-1}(t))}\right)\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)$$
$$=\textstyle\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.12}$$

Since $g(t)$ is an invertible series and $f(t)$ is a delta series of t therefore $\frac{g'\left(f^{-1}(t)\right)}{g(f^{-1}(t))}$ and $\frac{1}{f'(f^{-1}(t))}$ possess power series expansions of $f^{-1}(t)$. Thus, in view of relation (2.10), the above equation becomes

$$\left(\left(\hat{M}_{LH}-\frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\right)\left\{\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)\right\}$$
$$=\textstyle\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_{n+1}(x,y,z)\frac{t^n}{n!}, \tag{2.13}$$

which on using generating function (2.1) becomes

$$\left(\left(\hat{M}_{LH}-\frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{1}{f'(\partial_y)}\right)\left\{\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_n(x,y,z)\frac{t^n}{n!}\right\}=\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_{n+1}(x,y,z)\frac{t^n}{n!},$$

or, equivalently

$$\sum_{n=0}^{\infty}\left(\left(\hat{M}_{LH}-\frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{1}{f'(\partial_y)}\right)\left\{{}_{L}H^{(m,r)}s_n(x,y,z)\right\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_{L}H^{(m,r)}s_{n+1}(x,y,z)\frac{t^n}{n!}. \tag{2.14}$$

Now, equating the coefficients of like powers of t in the above equation, we find

$$\left(\left(\hat{M}_{LH}-\frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{1}{f'(\partial_y)}\right)\left\{{}_{L}H^{(m,r)}s_n(x,y,z)\right\}={}_{L}H^{(m,r)}\ s_{n+1}(x,y,z), \tag{2.15}$$

which, in view of equation (1.7) shows that the multiplicative operator for ${}_{L}H^{(m,r)}s_n(x,y,z)$ is given as:

$$\hat{M}_{LHs}=\left(\hat{M}_{LH}-\frac{g'\left(\partial_y\right)}{g\left(\partial_y\right)}\right)\frac{1}{f'(\partial_y)}.$$

Finally, using equation (1.17) in the r.h.s of above equation, we get assertion (2.8a).

Again, in view of identity (2.11), we have

$$f\left(\partial_y\right)\left\{\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)\right\}$$
$$=t\frac{1}{g\big(f^{-1}(t)\big)}C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right),$$

which on using generating function (2.1) becomes

$$f(\partial_y)\left\{\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{LH^{(m,r)}}s_{n-1}(x,y,z)\frac{t^n}{(n-1)!},$$

or, equivalently

$$\sum_{n=0}^{\infty} f(\partial_y)\left\{{}_{LH^{(m,r)}}s_n(x,y,z)\right\}\frac{t^n}{n!} = \sum_{n=1}^{\infty} {}_{LH^{(m,r)}}s_{n-1}(x,y,z)\frac{t^n}{(n-1)!}.$$

Equating the coefficients of like powers of t in the above equation, we get

$$f(\partial_y)\left\{{}_{LH^{(m,r)}}s_n(x,y,z)\right\} = n\ {}_{LH^{(m,r)}}s_{n-1}(x,y,z), \qquad n \geq 1, \tag{2.16}$$

which in view of equation (1.8) yields assertion (2.9a). Also, in view of relations (1.6a) and (1.6b), assertions (2.8a) and (2.9a) can be expressed equivalently as equations (2.8b) and (2.9b), respectively.

Remark 2.1. In view of equation (1.11) and using equations (2.8a) and (2.8b), we deduce the following consequence of Theorem 2.2.

Corollary 2.1. *The Laguerre-Gould Hopper based Sheffer polynomials* ${}_{LH^{(m,r)}}s_n(x,y,z)$ *have the following explicit representations:*

$${}_{LH^{(m,r)}}s_n(x,y,z) = \hat{M}^n_{LHs}\{1\}$$

that is,

$${}_{LH^{(m,r)}}s_n(x,y,z) = \left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\right)^n\{1\}, \tag{2.17a}$$

or, equivalently,

$${}_{LH^{(m,r)}}s_n(x,y,z) = \left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)^n\{1\}. \tag{2.17b}$$

Theorem 2.3. *The Laguerre-Gould Hopper based Sheffer polynomials* ${}_{LH^{(m,r)}}s_n(x,y,z)$ *satisfy the following differential equation:*

$$\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{f(\partial_y)}{f'(\partial_y)} - n\right){}_{LH^{(m,r)}}s_n(x,y,z) = 0, \tag{2.18a}$$

or, equivalently

$$\left(\left(\left(y + mD_x^{-1}\tfrac{\partial^{m-1}}{\partial y^{m-1}} + rz\tfrac{\partial^{r-1}}{\partial y^{r-1}}\right)H'\left(H^{-1}(\partial_y)\right) + \tfrac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)H^{-1}(\partial_y) - n\right) {}_{LH^{(m,r)}}s_n(x,y,z) = 0. \tag{2.18b}$$

Proof. Using equations (2.8a) and (2.9a) in equation (1.10), we get assertion (2.18a) and similarly using equations (2.8b) and (2.9b) in equation (1.10), we get assertion (2.18b).

Remark 2.2. Since the Sheffer sequence $s_n(x)$ for $g(t) = 1$ becomes the associated Sheffer sequence $\mathfrak{s}_n(x)$ for $f(t)$. (For our convenience, we denote the associated Sheffer sequence by $\mathfrak{s}_n(x)$). Therefore, for $g(t) = 1$, we deduce the following consequences of Theorems 2.1-2.3 :

Corollary 2.2. *The Laguerre-Gould Hopper based associated Sheffer polynomials (LGHASP)* ${}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)$ *are defined by the generating function*

$$C_0\left(-x\big(f^{-1}(t)\big)^m\right)\exp\left(yf^{-1}(t)+z\big(f^{-1}(t)\big)^r\right)=\sum_{n=0}^{\infty}{}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!}, \tag{2.19}$$

or, equivalently

$$C_0\left(-x\big(H(t)\big)^m\right)\exp\left(yH(t)+z\big(H(t)\big)^r\right)=\sum_{n=0}^{\infty}{}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)\frac{t^n}{n!}. \tag{2.20}$$

Corollary 2.3. *The Laguerre-Gould Hopper based associated Sheffer polynomials* ${}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)$ *are quasi-monomial under the action of the following multiplicative and derivative operators:*

$$\hat{M}_{LH\mathfrak{s}}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)\frac{1}{f'\left(\partial_y\right)}, \tag{2.21a}$$

or, equivalently

$$\hat{M}_{LH\mathfrak{s}}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)H'\left(H^{-1}\left(\partial_y\right)\right) \tag{2.21b}$$

and

$$\hat{P}_{LH\mathfrak{s}}=f\left(\partial_y\right), \tag{2.22a}$$

or, equivalently

$$\hat{P}_{LH\mathfrak{s}}=H^{-1}\left(\partial_y\right), \tag{2.22b}$$

respectively.

Corollary 2.4. *The Laguerre-Gould Hopper based associated Sheffer polynomials* ${}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)$ *satisfy the following differential equation:*

$$\left(\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)\frac{f\left(\partial_y\right)}{f'\left(\partial_y\right)}-n\right){}_LH^{(m,r)}\mathfrak{s}_n(x,y,z)=0, \tag{2.23a}$$

or, equivalently

$$\left(\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} + rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right) H'\left(H^{-1}\left(\partial_y\right)\right)\right) H^{-1}\left(\partial_y\right) - n\right) {}_{LH^{(m,r)}}\mathfrak{s}_n(x,y,z) = 0. \tag{2.23b}$$

Remark 2.3. Since, for $f(t) = t$, the Sheffer polynomials $s_n(x)$ reduce to the Appell polynomials $A_n(x)$ [2]. Therefore, taking $f(t) = t$ in Theorems 2.1-2.3, we obtain the corresponding results for the Laguerre-Gould Hopper based Appell polynomials (LGHAP) ${}_{LH^{(m,r)}}A_n(x,y,z)$ [34].

In the next section, we consider certain new and known families of special polynomials related to the Sheffer sequences and obtain the results for these mixed type special polynomials.

3. Examples

In Table 1, we have mentioned special cases of the LGHP ${}_LH_n^{(m,r)}(x,y,z)$. In order to obtain the results for the corresponding new or known special polynomials related to the Sheffer sequences, we consider the following examples:

Example 1. Since, for $m = 1$, $r = 2$, $x \to -x$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ (Table 1(I)). Therefore, for the same choice of m, r and x, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the 3-variable Laguerre-Hermite based Sheffer polynomials (3VLHSP) ${}_{LH}s_n(x,y,z)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the 3VLHSP ${}_{LH}s_n(x,y,z)$:

Table 2. Results for the 3VLHSP ${}_{LH}s_n(x,y,z)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))} C_0\left(x(f^{-1}(t))\right) \exp\left(yf^{-1}(t) + z(f^{-1}(t))^2\right)$ $= A(t)C_0\left(x(H(t))\right) \exp\left(yH(t) + z(H(t))^2\right) = \sum_{n=0}^{\infty} {}_{LH}s_n(x,y,z)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} = \left(y - D_x^{-1} + 2z\frac{\partial}{\partial y} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}$ $= \left(y - D_x^{-1} + 2z\frac{\partial}{\partial y}\right) H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}, \quad \hat{P} = f(\partial_y) = H^{-1}(\partial_y)$
3.	Differential equation	$\left(\left(y - D_x^{-1} + 2z\frac{\partial}{\partial y} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{f(\partial_y)}{f'(\partial_y)} - n\right) {}_{LH}s_n(x,y,z) = 0$, or equivalently $\left(\left(\left(y - D_x^{-1} + 2z\frac{\partial}{\partial y}\right) H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right) H^{-1}(\partial_y) - n\right) {}_{LH}s_n(x,y,z) = 0$
4.	Explicit representation	${}_{LH}s_n(x,y,z) = \left(\left(y - D_x^{-1} + 2z\frac{\partial}{\partial y} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\right)^n\{1\}$ $= \left(\left(y - D_x^{-1} + 2z\frac{\partial}{\partial y}\right) H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)^n\{1\}$

Example 2. Since, for $m = 1$, $r = 2$, $x \to -x$, $z = -\frac{1}{2}$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable Laguerre-Hermite polynomials (2VLHP) ${}_LH_n^{\star}(x,y)$ (Table 1(II)). Therefore, for the same choice of m, r, x and z, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable Laguerre-Hermite based Sheffer polynomials (2VLHSP) ${}_{LH^{\star}}s_n(x,y)$. Thus, by using these substitutions in Theorems

2.1, 2.2 and 2.3, we get the following results for the 2VLHSP ${}_{LH^\star}s_n(x,y)$:

Table 3. Results for the 2VLHSP ${}_{LH^\star}s_n(x,y)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(x(f^{-1}(t))\right)\exp\left(yf^{-1}(t)-\frac{1}{2}(f^{-1}(t))^2\right)$ $= A(t)C_0\left(x(H(t))\right)\exp\left(yH(t)-\frac{1}{2}(H(t))^2\right)=\sum_{n=0}^{\infty}{}_{LH^\star}s_n(x,y)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}=\left(y-D_x^{-1}-\frac{\partial}{\partial y}-\frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}$ $=\left(y-D_x^{-1}-\frac{\partial}{\partial y}\right)H'\left(H^{-1}(\partial_y)\right)+\frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}$, $\hat{P}=f(\partial_y)=H^{-1}(\partial_y)$
3.	Differential equation	$\left(\left(y-D_x^{-1}-\frac{\partial}{\partial y}-\frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{f(\partial_y)}{f'(\partial_y)}-n\right){}_{LH^\star}s_n(x,y)=0$, or equivalently $\left(\left(\left(y-D_x^{-1}-\frac{\partial}{\partial y}\right)H'\left(H^{-1}(\partial_y)\right)+\frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)H^{-1}(\partial_y)-n\right){}_{LH^\star}s_n(x,y)=0$
4.	Explicit representation	${}_{LH^\star}s_n(x,y)=\left(\left(y-D_x^{-1}-\frac{\partial}{\partial y}-\frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\right)^n\{1\}$ $=\left(\left(y-D_x^{-1}-\frac{\partial}{\partial y}\right)H'\left(H^{-1}(\partial_y)\right)+\frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)^n\{1\}$

Example 3. Since, for $m=1$, $r=2$, $x\to -x$, $y=1$, $z\to y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the Laguerre-Hermite type polynomials (LHTP) $\varphi_n(x,y)$ (Table 1(III)). Therefore, for the same choice of m, r, x, y and z, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable Laguerre-Hermite type based Sheffer polynomials (LHTSP) ${}_{\varphi}s_n(x,y)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the 2VLHTSP ${}_{\varphi}s_n(x,y)$:

Table 4. Results for the 2VLHTSP ${}_{\varphi}s_n(x,y)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(x(f^{-1}(t))\right)\exp\left(f^{-1}(t)+y(f^{-1}(t))^2\right)$ $= A(t)C_0\left(x(H(t))\right)\exp\left(H(t)+y(H(t))^2\right)=\sum_{n=0}^{\infty}{}_{\varphi}s_n(x,y)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}=\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{1}{f'(-\partial_x x\partial_x)}$ $=\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right)+\frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}$, $\hat{P}=f(-\partial_x x\partial_x)=H^{-1}(-\partial_x x\partial_x)$
3.	Differential equation	$\left(\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{f(-\partial_x x\partial_x)}{f'(-\partial_x x\partial_x)}-n\right){}_{\varphi}s_n(x,y)=0$, or equivalently $\left(\left(\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right)+\frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}\right)H^{-1}(-\partial_x x\partial_x)-n\right){}_{\varphi}s_n(x,y)=0$
4.	Explicit representation	${}_{\varphi}s_n(x,y)=\left(\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{1}{f'(-\partial_x x\partial_x)}\right)^n\{1\}$ $=\left(\left(1-D_x^{-1}-2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right)+\frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}\right)^n\{1\}$

Example 4. Since, for $x=0$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the Gould Hopper polynomials (GHP) $H_n^{(r)}(y,z)$ (Table 1(IV)). Therefore, for the same choice of x, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the Gould Hopper based Sheffer polynomials (GHSP) ${}_{H^{(r)}}s_n(y,z)$ [35]. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we obtain the corresponding results for the GHSP ${}_{H^{(r)}}s_n(y,z)$ [35].

Example 5. Since, for $z=0$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable generalized

Laguerre polynomials (2VLP) ${}_mL_n(x,y)$ (Table 1(V)). Therefore, for the same choice of z, the LGHSP ${}_{_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable generalized Laguerre based Sheffer polynomials (2VGLSP) ${}_{_mL}s_n(x,y)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the 2VGLSP ${}_{_mL}s_n(x,y)$:

Table 5. Results for the 2VGLSP ${}_{_mL}s_n(x,y)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(-x(f^{-1}(t))^m\right)\exp\left(yf^{-1}(t)\right) = A(t)C_0\left(-x(H(t))^m\right)\exp\left(yH(t)\right) = \sum_{n=0}^{\infty} {}_{_mL}s_n(x,y)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} = \left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)} = \left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}, \quad \hat{P} = f(\partial_y) = H^{-1}(\partial_y)$
3.	Differential equation	$\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{f(\partial_y)}{f'(\partial_y)} - n\right){}_{_mL}s_n(x,y) = 0$, or equivalently $\left(\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)H^{-1}(\partial_y) - n\right){}_{_mL}s_n(x,y) = 0$
4.	Explicit representation	${}_{_mL}s_n(x,y) = \left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}} - \frac{g'(\partial_y)}{g(\partial_y)}\right)\frac{1}{f'(\partial_y)}\right)^n\{1\} = \left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)H'\left(H^{-1}(\partial_y)\right) + \frac{A'\left(H^{-1}(\partial_y)\right)}{A\left(H^{-1}(\partial_y)\right)}\right)^n\{1\}$

Example 6. Since, for $r = m$, $x = 0$, $y \to -D_x^{-1}$, $z \to y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable generalized Laguerre type polynomials (2VGLTP) ${}_{[m]}L_n(x,y)$ (Table 1(VI)). Therefore, for the same choice of r, x, y and z, the LGHSP ${}_{_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable generalized Laguerre type based Sheffer polynomials (2VGLTSP) ${}_{_{[m]}L}s_n(x,y)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the 2VGLTSP ${}_{_{[m]}L}s_n(x,y)$:

Table 6. Results for the 2VGLTSP ${}_{_{[m]}L}s_n(x,y)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(xf^{-1}(t)\right)\exp\left(y(f^{-1}(t))^m\right) = A(t)C_0\left(xH(t)\right)\exp\left(y(H(t))^m\right) = \sum_{n=0}^{\infty} {}_{_{[m]}L}s_n(x,y)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} = \left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}} - \frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{1}{f'(-\partial_x x\partial_x)} = \left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}$, $\hat{P} = f(-\partial_x x\partial_x) = H^{-1}(-\partial_x x\partial_x)$
3.	Differential equation	$\left(\left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}} - \frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{f(-\partial_x x\partial_x)}{f'(-\partial_x x\partial_x)} - n\right){}_{_{[m]}L}s_n(x,y) = 0$, or $\left(\left(\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}\right)H^{-1}(-\partial_x x\partial_x) - n\right){}_{_{[m]}L}s_n(x,y) = 0$
4.	Explicit representation	${}_{_{[m]}L}s_n(x,y) = \left(\left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}} - \frac{g'(-\partial_x x\partial_x)}{g(-\partial_x x\partial_x)}\right)\frac{1}{f'(-\partial_x x\partial_x)}\right)^n\{1\} = \left(\left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)H'\left(H^{-1}(-\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(-\partial_x x\partial_x)\right)}{A\left(H^{-1}(-\partial_x x\partial_x)\right)}\right)^n\{1\}$

Example 7. Since, for $r = m-1$, $x = 0$, $y \to x$, $z \to y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the generalized Chebyshev polynomials (GCP) $U_n^{(m)}(x,y)$ (Table 1(VII)). Therefore, for the same choice of r, x, y and z, the LGHSP ${}_{_LH^{(m,r)}}s_n(x,y,z)$ reduce to the generalized Chebyshev based Sheffer polynomials (GCSP) ${}_{U^{(m)}}s_n(x,y)$ [30]. Thus, by using these substitutions in Theorems 2.1,

2.2 and 2.3, we obtain the corresponding results for the GCSP ${}_{U^{(m)}}s_n(x,y)$ [30].

Example 8. Since, for $m=1$, $x \to -x$, $z=0$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable Laguerre polynomials (2VLP) $L_n(x,y)$ (Table 1(VIII)). Therefore, for the same choice of m, x and z, the LGHSP ${}_{{}_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable Laguerre based Sheffer polynomials (2VLSP) ${}_Ls_n(x,y)$ [32]. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we obtain the corresponding results for the 2VLSP ${}_Ls_n(x,y)$ [32].

Example 9. Since, for $m=1$, $x \to y$, $y \to -D_x^{-1}$, $z=0$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable Legendre polynomials (2VLeP) $\frac{R_n(x,y)}{n!}$ (Table 1(IX)). Therefore, for the same choice of m, x, y and z, the LGHSP ${}_{{}_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable Legendre based Sheffer polynomials (2VLeSP) $\frac{{}_Rs_n(x,y)}{n!}$ [33]. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we obtain the corresponding results for the 2VLeSP $\frac{{}_Rs_n(x,y)}{n!}$ [33].

Example 10. Since, for $x=0$, $y \to x$, $z \to y\partial_y y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable truncated polynomials of order r (2VTP) $e_n^{(r)}(x,y)$ (Table 1(X)). Therefore, for the same choice of x, y and z, the LGHSP ${}_{{}_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable truncated exponential based Sheffer polynomials (2VTESP) ${}_{e^{(r)}}s_n(x,y)$ [36]. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we obtain the corresponding results for the 2VTESP ${}_{e^{(r)}}s_n(x,y)$ [36].

Example 11. Since, for $r=2$, $x=0$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(y,z)$ (Table 1(XI)). Therefore, for the same choice of r and x, the LGHSP ${}_{{}_LH^{(m,r)}}s_n(x,y,z)$ reduce to the 2-variable Hermite Kampé de Fériet based Sheffer polynomials (2VHKdFSP) ${}_Hs_n(y,z)$ [31]. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we obtain the corresponding results for the 2VHKdFSP ${}_Hs_n(y,z)$ [31].

Example 12. Since, for $r=2$, $x=0$, $y \to D_x^{-1}$, $z \to y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the Hermite type polynomials (HTP) $G_n(x,y)$ (Table 1(XII)). Therefore, for the same choice of r, x, y and z, the LGHSP ${}_{{}_LH^{(m,r)}}s_n(x,y,z)$ reduce to the Hermite type based Sheffer polynomials (HTSP) ${}_Gs_n(x,y)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the HTSP ${}_Gs_n(x,y)$:

Table 7. Results for the HTSP ${}_{G}s_n(x,y)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(-xf^{-1}(t)\right)\exp\left(y(f^{-1}(t))^2\right)$ $= A(t)C_0\left(-xH(t)\right)\exp\left(y(H(t))^2\right) = \sum_{n=0}^{\infty} {}_{G}s_n(x,y)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} = \left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x} - \frac{g'(\partial_x x\partial_x)}{g(\partial_x x\partial_x)}\right)\frac{1}{f'(\partial_x x\partial_x)}$ $= \left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x x\partial_x)\right)}{A\left(H^{-1}(\partial_x x\partial_x)\right)}$, $\hat{P} = f(\partial_x x\partial_x) = H^{-1}(\partial_x x\partial_x)$
3.	Differential equation	$\left(\left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x} - \frac{g'(\partial_x x\partial_x)}{g(\partial_x x\partial_x)}\right)\frac{f(\partial_x x\partial_x)}{f'(\partial_x x\partial_x)} - n\right){}_{G}s_n(x,y) = 0$, or equivalently $\left(\left(\left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x x\partial_x)\right)}{A\left(H^{-1}(\partial_x x\partial_x)\right)}\right)H^{-1}(\partial_x x\partial_x) - n\right){}_{G}s_n(x,y) = 0$
4.	Explicit representation	${}_{G}s_n(x,y) = \left(\left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x} - \frac{g'(\partial_x x\partial_x)}{g(\partial_x x\partial_x)}\right)\frac{1}{f'(\partial_x x\partial_x)}\right)^n\{1\}$ $= \left(\left(D_x^{-1} + 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x x\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x x\partial_x)\right)}{A\left(H^{-1}(\partial_x x\partial_x)\right)}\right)^n\{1\}$

Example 13. Since, for $m=2$, $x\to\left(\frac{x^2-1}{4}\right)$, $y\to x$, $z=0$, the LGHP ${}_{L}H_n^{(m,r)}(x,y,z)$ reduce to the Legendre polynomials (LeP) $P_n(x)$ (Table 1(XIII)). Therefore for the same substitutions of m, x, y and z, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the Legendre based Sheffer polynomials (LeSP) ${}_{P}s_n(x)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the LeSP ${}_{P}s_n(x)$:

Table 8. Results for the LeSP ${}_{P}s_n(x)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}C_0\left(-\left(\frac{x^2-1}{4}\right)(f^{-1}(t))^2\right)\exp\left(xf^{-1}(t)\right)$ $= A(t)C_0\left(-\left(\frac{x^2-1}{4}\right)(H(t))^2\right)\exp\left(xH(t)\right) = \sum_{n=0}^{\infty} {}_{P}s_n(x)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M} = \left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x} - \frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}$ $= \left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}$, $\hat{P} = f(\partial_x) = H^{-1}(\partial_x)$
3.	Differential equation	$\left(\left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x} - \frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{f(\partial_x)}{f'(\partial_x)} - n\right){}_{P}s_n(x) = 0$, or equivalently $\left(\left(\left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)H^{-1}(\partial_x) - n\right){}_{P}s_n(x) = 0$
4.	Explicit representation	${}_{P}s_n(x) = \left(\left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x} - \frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}\right)^n\{1\}$ $= \left(\left(x + 2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right) + \frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)^n\{1\}$

Example 14. Since, for $x\to y\partial_y y$, $y\to x$, the LGHP ${}_{L}H_n^{(m,r)}(x,y,z)$ reduce to the 3-variable generalized Hermite polynomials (3VGHP) $H_n^{(r,m)}(x,y,z)$ (Table 1(XIV)). Therefore, for the same choice of x and y, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the 3-variable generalized Hermite based

Sheffer polynomials (3VGHSP) ${}_{H^{(r,m)}}s_n(x,y,z)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the 3VGHSP ${}_{H^{(r,m)}}s_n(x,y,z)$:

Table 9. Results for the 3VGHSP ${}_{H^{(r,m)}}s_n(x,y,z)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)+y(f^{-1}(t))^m+z(f^{-1}(t))^r\right)$ $= A(t)\exp\left(xH(t)+y(H(t))^m+z(H(t))^r\right)=\sum_{n=0}^{\infty}{}_{H^{(r,m)}}s_n(x,y,z)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}=\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}$ $=\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}$, $\hat{P}=f(\partial_x)=H^{-1}(\partial_x)$
3.	Differential equation	$\left(\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{f(\partial_x)}{f'(\partial_x)}-n\right){}_{H^{(r,m)}}s_n(x,y,z)=0$, or equivalently $\left(\left(\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)H^{-1}(\partial_x)-n\right){}_{H^{(r,m)}}s_n(x,y,z)=0$
4.	Explicit representation	${}_{H^{(r,m)}}s_n(x,y,z)=\left(\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}\right)^n\{1\}$ $=\left(\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)^n\{1\}$

Example 15. Since, for $m=2$, $r=3$, $x\to z\partial_z z$, $y\to x$, $z\to y$, the LGHP ${}_LH_n^{(m,r)}(x,y,z)$ reduce to the Bell type polynomials (BTP) $H_n^{(3,2)}(x,y,z)$ (Table 1(XV)). Therefore, for the same choice of m, r, x, y and z, the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ reduce to the Bell type based Sheffer polynomials (BTSP) ${}_{H^{(3,2)}}s_n(x,y,z)$. Thus, by using these substitutions in Theorems 2.1, 2.2 and 2.3, we get the following results for the BTSP ${}_{H^{(3,2)}}s_n(x,y,z)$:

Table 10. Results for the BTSP ${}_{H^{(3,2)}}s_n(x,y,z)$

S. No.	Results	Mathematical Expressions
1.	Generating function	$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t)+y(f^{-1}(t))^3+z(f^{-1}(t))^2\right)$ $=A(t)\exp\left(xH(t)+y(H(t))^3+z(H(t))^2\right)=\sum_{n=0}^{\infty}{}_{H^{(3,2)}}s_n(x,y,z)\frac{t^n}{n!}$
2.	Multiplicative and derivative operators	$\hat{M}=\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}$ $=\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}$, $\hat{P}=f(\partial_x)=H^{-1}(\partial_x)$
3.	Differential equation	$\left(\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{f(\partial_x)}{f'(\partial_x)}-n\right){}_{H^{(3,2)}}s_n(x,y,z)=0$, or equivalently $\left(\left(\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)H^{-1}(\partial_x)-n\right){}_{H^{(3,2)}}s_n(x,y,z)=0$
4.	Explicit representation	${}_{H^{(3,2)}}s_n(x,y,z)=\left(\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}-\frac{g'(\partial_x)}{g(\partial_x)}\right)\frac{1}{f'(\partial_x)}\right)^n\{1\}$ $=\left(\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}\right)H'\left(H^{-1}(\partial_x)\right)+\frac{A'\left(H^{-1}(\partial_x)\right)}{A\left(H^{-1}(\partial_x)\right)}\right)^n\{1\}$

Remark 3.1. For $\frac{1}{g(f^{-1}(t))}=A(t)=1$, the above mentioned special cases (Examples 1-15) of the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ yield the corresponding results for the LGHASP ${}_{LH^{(m,r)}}\mathfrak{s}_n(x,y,z)$, which can also be deduced by using certain substitutions in Corollaries 2.2-2.4.

Remark 3.2. For $f^{-1}(t)=H(t)=t$, the above mentioned special cases (Examples 1-15) of the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ yield the corresponding results for the LGHAP ${}_{LH^{(m,r)}}A_n(x,y,z)$ [34].

In the next section, we derive certain operational and integral representations for the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$.

4. Operational and integral representations

To establish the operational representations for the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$, we prove the following results:

Theorem 4.1. *The following operational representation connecting the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ with the Sheffer polynomials $s_n(x)$ holds true:*

$$ {}_{LH^{(m,r)}}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m} + z\frac{\partial^r}{\partial y^r}\right) s_n(y) \tag{4.1} $$

Proof. In view of equation (2.7), the proof is direct use of identity (2.4).

Theorem 4.2. *The following operational representation connecting the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ with the 2VGLSP ${}_{mL}s_n(x,y)$ holds true:*

$$ {}_{LH^{(m,r)}}s_n(x,y,z) = \exp\left(z\frac{\partial^r}{\partial y^r}\right) {}_{mL}s_n(x,y) \tag{4.2} $$

Proof. From equation (2.1) (or (2.2)), we have

$$ \frac{\partial^r}{\partial y^r}\, {}_{LH^{(m,r)}}s_n(x,y,z) = \frac{\partial}{\partial z}\, {}_{LH^{(m,r)}}s_n(x,y,z). \tag{4.3} $$

Since, in view of Table 1(V), we have

$$ {}_{L}H_n^{(m,r)}(x,y,0) = {}_{m}L_n(x,y). \tag{4.4} $$

Therefore, from Example 5 of Section 3, we have

$$ {}_{LH^{(m,r)}}s_n(x,y,0) = {}_{mL}\, s_n(x,y). \tag{4.5} $$

Now, solving equations (4.3) subject to initial condition (4.5), we get assertion (4.2).

Theorem 4.3. *The following operational representation connecting the LGHSP ${}_{LH^{(m,r)}}s_n(x,y,z)$ with the GHSP ${}_{H^{(r)}}s_n(y,z)$ holds true:*

$$ {}_{LH^{(m,r)}}s_n(x,y,z) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right) {}_{H^{(r)}}s_n(y,z). \tag{4.6} $$

Proof. From equations (1.15) and (2.1) (or (2.2)), we have

$$ \frac{\partial^m}{\partial y^m}\, {}_{LH^{(m.r)}}s_n(x,y,z) = \frac{\partial}{\partial D_x^{-1}}\, {}_{LH^{(m,r)}}s_n(x,y,z), \tag{4.7} $$

where ([19]; p. 32 (8)):

$$\frac{\partial}{\partial D_x^{-1}} := \frac{\partial}{\partial x} x \frac{\partial}{\partial x}.$$

Since, in view of Table 1(IV), we have

$$_L H_n^{(m,r)}(0,y,z) = H^{(r)}(y,z). \tag{4.8}$$

Therefore, from Example 4 of Section 3, we have

$$_{L}H^{(m,r)}s_n(0,y,z) = _{H^{(r)}}\ s_n(y,z). \tag{4.9}$$

Solving equation (4.7) subject to initial condition (4.9), we get assertion (4.6).

By making suitable substitutions for the indices and variables in operational representation (4.1), we find the operational representations for the special cases of the LGHSP $_{L}H^{(m,r)}s_n(x,y,z)$. We mention these operational representations in Table 11.

Table 11. Operational representations for the special cases of the LGHSP $_{L}H^{(m,r)}s_n(x,y,z)$

S. No.	Values of the Indices and Variables	Relation Between the LGHSP $_{L}H^{(m,r)}s_n(x,y,z)$ and its Special Case	Name of the Polynomials	Operational Definition of the Polynomials
I.	$m=1,\ r=2;$ $x \to -x$	$_{L}H^{(1,2)}s_n(-x,y,z)$ $= _{LH}\ s_n(x,y,z)$	3-Variable Laguerre -Hermite based Sheffer	$_{LH}s_n(x,y,z) = s_n\left(y - D_x^{-1} + 2z\frac{\partial}{\partial y}\right)$ $= \exp\left(-D_x^{-1}\frac{\partial}{\partial y} + z\frac{\partial^2}{\partial y^2}\right)s_n(y)$
II.	$m=1,\ r=2;$ $z=-\frac{1}{2},$ $x \to -x$	$_{L}H^{(1,2)}s_n(-x,y,-\frac{1}{2})$ $= _{LH^*}\ s_n(x,y)$	2-Variable Laguerre -Hermite based Sheffer	$_{LH^\star}s_n(x,y) = s_n\left(y - D_x^{-1} - \frac{\partial}{\partial y}\right)$ $= \exp\left(-D_x^{-1}\frac{\partial}{\partial y} - \frac{1}{2}\frac{\partial^2}{\partial y^2}\right)s_n(y)$
III.	$m=1,\ r=2;$ $y=1,\ z \to y,$ $x \to -x$	$_{L}H^{(1,2)}s_n(-x,1,y)$ $= _{\varphi}\ s_n(x,y)$	Laguerre -Hermite type based Sheffer	$_{\varphi}s_n(x,y) = s_n\left(1 - D_x^{-1} - 2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)$ $= \exp\left(y\frac{\partial^2}{\partial x^2}x^2\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)s_n\left(-D_x^{-1}\right)$
IV.	$x=0$	$_{L}H^{(m,r)}s_n(0,y,z)$ $= _{H^{(r)}}\ s_n(y,z)$	Gould-Hopper based Sheffer	$_{H^{(r)}}s_n(y,z) = s_n\left(y + rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $= \exp\left(z\frac{\partial^r}{\partial y^r}\right)s_n(y)$ [35]
V.	$z=0$	$_{L}H^{(m,r)}s_n(x,y,0)$ $= _{mL_n}\ s_n(x,y)$	2-Variable generalized Laguerre based Sheffer	$_{mL}s_n(x,y) = s_n\left(y + mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}\right)$ $= \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)s_n(y)$
VI.	$r=m;\ x=0,$ $y \to -D_x^{-1},$ $z \to y$	$_{L}H^{(m,m)}s_n(0,-D_x^{-1},y)$ $= _{[m]L}\ s_n(x,y)$	2-Variable generalized Laguerre type based Sheffer	$_{[m]L}s_n(x,y)$ $= s_n\left(-D_x^{-1} + (-1)^m my\frac{\partial^{m-1}}{\partial x^{m-1}}x^{m-1}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)$ $= \exp\left(-(-1)^m y\frac{\partial^m}{\partial x^m}x^m\frac{\partial^m}{\partial x^m}\right)s_n(-D_x^{-1})$
VII.	$r=m-1;\ x=0,$ $y \to x,\ z \to y$	$_{L}H^{(m,m-1)}s_n(0,x,y)$ $= _{U^{(m)}}\ s_n(x,y)$	Generalized Chebyshev based Sheffer	$_{U^{(m)}}s_n(x,y) = s_n\left(x + (m-1)y\frac{\partial^{m-2}}{\partial x^{m-2}}\right)$ $=\exp\left(y\frac{\partial^{m-1}}{\partial x^{m-1}}\right)s_n(x)$
VIII.	$m=1;\ z=0,$ $x \to -x$	$_{L}H^{(1,r)}s_n(-x,y,0)$ $= _{L}\ s_n(x,y)$	2-Variable Laguerre based Sheffer	$_{L}s_n(x,y) = s_n\left(y - D_x^{-1}\right)$ $= \exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right)s_n(y)$ $= \exp\left(-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)s_n(-D_x^{-1})$ [32]
IX.	$m=1;\ z=0,$ $x \to y,$ $y \to -D_x^{-1}$	$n!\ _{L}H^{(1,r)}s_n(y,-D_x^{-1},0)$ $= _{R}\ s_n(x,y)$	2-Variable Legendre based Sheffer	$\frac{_{R}s_n(x,y)}{n!} = s_n\left(-D_x^{-1} + D_y^{-1}\right)$ $= \exp\left(-D_x^{-1}\frac{\partial}{\partial y}y\frac{\partial}{\partial y}\right)s_n(D_y^{-1})$ $= \exp\left(-D_y^{-1}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)s_n(-D_x^{-1})$ [33]

X.	$x=0,\ y\to x,$ $z\to yD_y y$	${}_LH^{(m,r)}s_n(0,x,yD_y y)$ $={}_{e^{(r)}}s_n(x,y)$	2-Variable truncated of order r based Sheffer	${}_{e^{(r)}}s_n(x,y)=s_n\left(x+ry\frac{\partial}{\partial y}y\frac{\partial^{r-1}}{\partial x^{r-1}}\right)$ $=\exp\left(y\frac{\partial}{\partial y}y\frac{\partial^r}{\partial x^r}\right)s_n(x)$ [36]
XI.	$r=2;\ x=0$	${}_LH^{(m,2)}s_n(0,y,z)$ $={}_Hs_n(y,z)$	2-Variable Hermite- Kampé de Fériet based Sheffer	${}_Hs_n(y,z)=s_n\left(y+2z\frac{\partial}{\partial y}\right)$ $=\exp\left(z\frac{\partial^2}{\partial y^2}\right)s_n(y)$ [31]
XII.	$r=2;\ x=0,$ $y\to D_x^{-1},\ z\to y$	${}_LH^{(m,2)}s_n(0,D_x^{-1},y)$ $={}_Gs_n(x,y)$	Hermite type based Sheffer	${}_Gs_n(x,y)=s_n\left(D_x^{-1}+2y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)$ $=\exp\left(y\frac{\partial^2}{\partial x^2}x^2\frac{\partial^2}{\partial x^2}\right)s_n(D_x^{-1})$
XIII.	$m=2;\ z=0,$ $x\to(\frac{x^2-1}{4}),$ $y\to x,$	${}_LH^{(2,r)}s_n(\frac{x^2-1}{4},x,0)$ $={}_Ps_n(x)$	Legendre based Sheffer	${}_Ps_n(x)=s_n\left(x+2D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial}{\partial x}\right)$ $=\exp\left(D^{-1}_{\left(\frac{x^2-1}{4}\right)}\frac{\partial^2}{\partial x^2}\right)s_n(x)$
XIV.	$x\to yD_y y,$ $y\to x$	${}_LH^{(m,r)}s_n(yD_y\ y,x,z)$ $={}_{H^{(r,m)}}s_n(x,y,z)$	3-Variable generalized Hermite based Sheffer	${}_{H^{(r,m)}}s_n(x,y,z)$ $=s_n\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}+rz\frac{\partial^{r-1}}{\partial x^{r-1}}\right)$ $=\exp\left(y\frac{\partial^m}{\partial x^m}+z\frac{\partial^r}{\partial x^r}\right)s_n(x)$
XV.	$m=2,\ r=3;$ $x\to zD_z z,$ $y\to x,\ z\to y$	${}_LH_n^{(2,3)}s_n(zD_z\ z,x,y)$ $={}_{H^{(3,2)}}s_n(x,y,z)$	Bell-type based Sheffer	${}_{H^{(3,2)}}s_n(x,y,z)=s_n\left(x+3y\frac{\partial^2}{\partial x^2}+2z\frac{\partial}{\partial x}\right)$ $=\exp\left(y\frac{\partial^3}{\partial x^3}+z\frac{\partial^2}{\partial x^2}\right)s_n(x)$

Next, we prove the integral representations for the LGHSP ${}_{L}H^{(m,r)}s_n(x,y,z)$ in the form of following theorems:

Theorem 4.4. *The following integral representation for the LGHSP* ${}_{L}H^{(m,r)}s_n(x,y,z)$ *holds true:*

$$ {}_{L}H^{(m,r)}s_n(x,y,z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^u\ u^{-1}\ {}_{H^{(r,m)}}s_n(y,xu^{-1},z)\ du\ . \tag{4.10}$$

Proof. Using equation (1.13) in the l.h.s. of equation (2.1) and interchanging the sides, we have

$$\sum_{n=0}^{\infty} {}_{L}H^{(m,r)}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{g\big(f^{-1}(t)\big)}\sum_{n=0}^{\infty} {}_{L}H_n^{(m,r)}(x,y,z)\frac{\big(f^{-1}(t)\big)^n}{n!}. \tag{4.11}$$

Using the following integral representation of LGHP ${}_{L}H_n^{(m,r)}(x,y,z)$ [29]:

$$ {}_{L}H_n^{(m,r)}(x,y,z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^u\ u^{-1}H_n^{(r,m)}(y,xu^{-1},z)\ du\ , \tag{4.12}$$

in the r.h.s. of equation (4.11), we find

$$\sum_{n=0}^{\infty} {}_{L}H^{(m,r)}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{2\pi i}\frac{1}{g\big(f^{-1}(t)\big)}\int_{-\infty}^{(0+)} e^u\ u^{-1}\left(\sum_{n=0}^{\infty}H_n^{(r,m)}(y,xu^{-1},z)\frac{\big(f^{-1}(t)\big)^n}{n!}\right)du\ . \tag{4.13}$$

Now, making use of the generating function equation of $H_n^{(r,m)}(x,y,z)$ given in Table 1(XIV) in the r.h.s. of the above equation, we have

$$\sum_{n=0}^{\infty} {}_{L}H^{(m,r)}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^u\ u^{-1}\left(\frac{1}{g\big(f^{-1}(t)\big)}e^{\left(yf^{-1}(t)+xu^{-1}\left(f^{-1}(t)\right)^r+z\left(f^{-1}(t)\right)^m\right)}\right)du\ ,$$

which in view of generating function equation of ${}_{H^{(r,m)}}s_n(x,y,z)$ given in Table 9 becomes

$$\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{-\infty}^{(0+)} e^u\ u^{-1}{}_{H^{(r,m)}}s_n(y,xu^{-1},z)\ du\right)\frac{t^n}{n!}. \tag{4.14}$$

Finally, equating the coefficients of like powers of t in both sides of equation (4.14), we get assertion (4.10).

Theorem 4.5. *The following integral representations for the LGHSP* ${}_{LH^{(m,r)}}s_n(x,y,z)$ *hold true:*

$$ {}_{LH^{(m,r)}}s_n(x,y,z) = \int_0^{\infty} e^{-u}\ {}_{LH^{(m,r)}}s_n(x,y,uD_z^{-1})\ du \tag{4.15}$$

and

$$ {}_{LH^{(m,r)}}s_n(x,y,z) = \frac{1}{n!}\int_0^{\infty} e^{-u}\ u^n\ {}_{LH^{(m+1,r)}}s_n\left(\frac{x}{u},y,z\right) du\ . \tag{4.16}$$

Proof. Using the integral representations [29]:

$$ {}_LH_n^{(m,r)}(x,y,z) = \int_0^{\infty} e^{-u}\ {}_LH_n^{(m,r)}(x,y,uD_z^{-1})\ du \tag{4.17}$$

and

$$ {}_LH_n^{(m,r)}(x,y,z) = \frac{1}{n!}\int_0^{\infty} e^{-u}\ u^n\ {}_LH_n^{(m+1,r)}\left(\frac{x}{u},y,z\right) du\ , \tag{4.18}$$

respectively in the r.h.s. of equation (4.11), we find

$$\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty}\left(\int_0^{\infty} e^{-u}\ {}_LH_n^{(m,r)}(x,y,uD_z^{-1})\ du\right)\frac{\left(f^{-1}(t)\right)^n}{n!},$$

or, equivalently

$$\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!} = \int_0^{\infty} e^{-u}\ \left(\frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty} {}_LH_n^{(m,r)}(x,y,uD_z^{-1})\frac{\left(f^{-1}(t)\right)^n}{n!}\right)\ du \tag{4.19}$$

and

$$\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty}\left(\frac{1}{n!}\int_0^{\infty} e^{-u}\ u^n\ {}_LH_n^{(m+1,r)}\left(\frac{x}{u},y,z\right) du\ \right)\frac{\left(f^{-1}(t)\right)^n}{n!},$$

or, equivalently

$$\sum_{n=0}^{\infty} {}_{LH^{(m,r)}}s_n(x,y,z)\frac{t^n}{n!} = \frac{1}{n!}\int_0^{\infty} e^{-u}\ u^n\ \left(\frac{1}{g(f^{-1}(t))}\sum_{n=0}^{\infty} {}_LH_n^{(m+1,r)}\left(\frac{x}{u},y,z\right)\frac{\left(f^{-1}(t)\right)^n}{n!}\right)\ du, \tag{4.20}$$

respectively.

Again, using equation (4.11) in the r.h.s. of equations (4.19) and (4.20), we find

$$\sum_{n=0}^{\infty} {}_LH^{(m,r)}s_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\int_0^{\infty} e^{-u} \ {}_LH^{(m,r)}s_n(x,y,uD_z^{-1})\ du\right)\frac{t^n}{n!} \tag{4.21}$$

and

$$\sum_{n=0}^{\infty} {}_LH^{(m,r)}s_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\int_0^{\infty} e^{-u}\ u^n \ {}_LH^{(m+1,r)}s_n\left(\frac{x}{u},y,z\right)du\right)\frac{t^n}{n!}, \tag{4.22}$$

respectively. Finally, equating the coefficients of like powers of t in both sides of equations (4.21) and (4.22), we get assertions (4.15) and (4.16), respectively.

In Section 3, we have obtained the results for the new and known families of special polynomials related to Sheffer sequences by taking the special cases of the LGHP ${}_LH_n^{(m,r)}(x,y,z)$. In the Appendix section, we consider certain special polynomials belonging to the Sheffer and associated Sheffer families and obtain the results for the corresponding mixed special polynomials.

5. Appendix

The Sheffer class contains important sequences such as the Hermite, Laguerre, Bernoulli, Poisson-Charlier polynomials *etc.* These polynomials are important from the view point of applications in physics and number theory. Also, the associated Sheffer family contains Mittag-Leffler, exponential, lower factorial polynomials *etc.*

We present the lists of some known members of the Sheffer and associated Sheffer families in Tables 12 and 13 respectively.

Table 12. Some members of the Sheffer family

S. No.	$g(t)$; $A(t)$	$f(t)$; $H(t)$	Generating Functions	Polynomials
I.	$e^{(\frac{t}{\nu})^k}$; e^{-t^k}	$\frac{t}{\nu}$; νt	$\exp(\nu x t - t^k) = \sum_{n=0}^{\infty} H_{n,k,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite polynomials $H_{n,k,\nu}(x)$ [38]
II.	$(1-t)^{-\alpha-1}$; $(1-t)^{-\alpha-1}$	$\frac{t}{t-1}$; $\frac{t}{t-1}$	$\frac{1}{(1-t)^{\alpha+1}}\exp(\frac{xt}{t-1}) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n$	Generalized Laguerre polynomials $n!L_n^{\alpha}(x)$ [1, 39]
III.	$\frac{2}{e^t-1}$; $\frac{t}{1-t}$	$\frac{e^t-1}{e^t+1}$; $\ln(\frac{1+t}{1-t})$	$\frac{t}{1-t}(\frac{1+t}{1-t})^x = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}$	Pidduck polynomials $P_n(x)$ [7, 26]
IV.	$(1-t)^{-\beta}$; $e^{\beta t}$	$\ln(1-t)$; $1-e^t$	$\exp(\beta t + x(1-e^t)) = \sum_{n=0}^{\infty} a_n^{(\beta)}(x)\frac{t^n}{n!}$	Acturial polynomials $a_n^{(\beta)}(x)$ [7]

V.	$\exp(a(e^t-1));\ e^{-t}$	$a(e^t-1);\ \ln(1+\frac{t}{a})$	$e^{-t}(1+\frac{t}{a})^x = \sum_{n=0}^{\infty} c_n(x;a)\frac{t^n}{n!}$	Poisson-Charlier polynomials $c_n(x;a)$ [25, 28, 42]
VI.	$(1+e^{\lambda t})^{\mu};\ (1+(1+t)^{\lambda})^{-\mu}$	$e^t-1;\ \ln(1+t)$	$(1+(1+t)^{\lambda})^{-\mu}(1+t)^x = \sum_{n=0}^{\infty} S_n(x;\lambda,\mu)\frac{t^n}{n!}$	Peters polynomials $S_n(x;\lambda,\mu)$ [7]
VII.	$\frac{t}{e^t-1};\ \frac{t}{\ln(1+t)}$	$e^t-1;\ \ln(1+t)$	$\frac{t}{\ln(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}$	Bernoulli polynomials of the second kind $b_n(x)$ [28]
VIII.	$\frac{1}{2}(1+e^t);\ \frac{2}{2+t}$	$e^t-1;\ \ln(1+t)$	$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}$	Related polynomials $r_n(x)$ [28]
IX.	$\sec t;\ \frac{1}{\sqrt{1+t^2}}$	$\tan t;\ \arctan(t)$	$\frac{1}{\sqrt{1+t^2}}\exp(x\arctan(t)) = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}$	Hahn polynomials $R_n(x)$ [6]
X.	$\frac{1+t}{(1-t)^a};\ (1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}$	$\frac{1}{4}-\frac{1}{4}\left(\frac{1+t}{1-t}\right)^2;\ \frac{-4t}{(1+\sqrt{1-4t})^2}$	$(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \times \exp\left(\frac{-4xt}{(1+\sqrt{1-4t})^2}\right) = \sum_{n=0}^{\infty} R_n(a,x)t^n$	Shively's psedo-Laguerre polynomials $R_n(a,x)$ [39]

Table 13. Some members of the associated Sheffer family

S. No.	$f(t);\ H(t)$	Generating Functions	Polynomials
I.	$\frac{e^t-1}{e^t+1};\ \ln\left(\frac{1+t}{1-t}\right)$	$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}$	Mittag-Leffler polynomials $M_n(x)$ [4]
II.	$\ln(1+t);\ e^t-1$	$\exp(x(e^t-1)) = \sum_{n=0}^{\infty} \varphi_n(x)\frac{t^n}{n!}$	Exponential polynomials $\varphi_n(x)$ [5]
III.	$e^t-1;\ \ln(1+t)$	$(1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}$	Lower factorial polynomials $(x)_n$ [40]
IV.	$-\frac{1}{2}t^2+t;\ 1-\sqrt{1-2t}$	$\exp(x(1-\sqrt{1-2t})) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}$	Bessel polynomials $p_n(x)$ [8, 37]

Remark 5.1. We remark that corresponding to each member belonging to the Sheffer (or associated Sheffer) family, there exists a new special polynomial belonging to the LGHSP (or LGHASP) family. The generating function and other properties of these special polynomials can be obtained from the results derived in Section 2.

Thus, by taking $g(t)$ (or $A(t)$) and $f(t)$ (or $H(t)$) of the special polynomials belonging to Sheffer family (Table 12 (I to X)) in equations (2.1) (or (2.2)), (2.8a) (or (2.8b)) and (2.9a) (or (2.9b)), we get the generating function, multiplicative and derivative operators for the corresponding members belonging to the LGHSP family.

We present these results along with the name and notation of the resultant special polynomial belonging to the LGHSP family in Table 14.

Table 14. Certain results for the members belonging to the LGHSP family

S. No.	Name/Notation of the Resultant Special Polynomial	Generating Function	Multiplicative and Derivative Operators
I.	${}_LH^{(m,r)}H_{n,k,\nu}(x,y,z):=$ Laguerre-Gould Hopper based generalized Hermite polynomials	$e^{-t^k}C_0(-x(\nu t)^m)\exp(y\nu t+z(\nu t)^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}H_{n,k,\nu}(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\nu\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $-k(\frac{1}{\nu}\partial_y)^{k-1}$, $\hat{P}=\frac{1}{\nu}\partial_y$
II.	$n!\,{}_LH^{(m,r)}L_n^{(\alpha)}(x,y,z):=$ Laguerre-Gould Hopper based generalized Laguerre polynomials	$(1-t)^{-\alpha-1}C_0\left(-x\left(\frac{t}{t-1}\right)^m\right)\exp\left(\frac{yt}{t-1}+z\left(\frac{t}{t-1}\right)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}L_n^{(\alpha)}(x,y,z)t^n$	$\hat{M}=-\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(\partial_y-1)^2+(a+1)(\partial_y-1)$, $\hat{P}=\frac{\partial_y}{\partial_y-1}$
III.	${}_LH^{(m,r)}P_n(x,y,z):=$ Laguerre-Gould Hopper based Pidduck polynomials	$\frac{t}{1-t}C_0\left(-x\left(\ln\left(\frac{1+t}{1-t}\right)\right)^m\right)\times$ $\exp\left(y\ln\left(\frac{1+t}{1-t}\right)+z\left(\ln\left(\frac{1+t}{1-t}\right)\right)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}P_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $\left(\frac{(e^{\partial_y}+1)^2}{2e^{\partial_y}}\right)+\frac{(e^{\partial_y}+1)^2}{2(e^{\partial_y}-1)}$, $\hat{P}=\frac{e^{\partial_y}-1}{e^{\partial_y}+1}$
IV.	${}_LH^{(m,r)}a_n^{(\beta)}(x,y,z):=$ Laguerre-Gould Hopper based Acturial polynomials	$e^{\beta t}C_0(-x(1-e^t)^m)\exp(y(1-e^t)+z(1-e^t)^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}a_n^{(\beta)}(x,y,z)\frac{t^n}{n!}$	$\hat{M}=-\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(1-\partial_y)+\beta$, $\hat{P}=\ln(1-\partial_y)$
V.	${}_LH^{(m,r)}c_n(x,y,z;a):=$ Laguerre-Gould Hopper based Poisson-Charlier polynomials	$e^{-t}C_0\left(-x\left(\ln\left(1+\frac{t}{a}\right)\right)^m\right)\times$ $\exp\left(y\ln\left(1+\frac{t}{a}\right)+z\left(\ln\left(1+\frac{t}{a}\right)\right)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}c_n(x,y,z;a)\frac{t^n}{n!}$	$\hat{M}=a^{-1}\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(e^{-\partial_y})-1$, $\hat{P}=a(e^{\partial_y}-1)$
VI.	${}_LH^{(m,r)}S_n(x,y,z;\lambda,\mu):=$ Laguerre-Gould Hopper based Peters polynomials	$(1+(1+t)^{\lambda})^{-\mu}C_0(-x(\ln(1+t))^m)\times$ $\exp(y\ln(1+t)+z(\ln(1+t))^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}S_n(x,y,z;\lambda,\mu)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(e^{-\partial_y})-\lambda\mu(1+e^{\lambda\partial_y})^{-1}e^{(\lambda-1)\partial_y}$, $\hat{P}=e^{\partial_y}-1$
VII.	${}_LH^{(m,r)}b_n(x,y,z):=$ Laguerre-Gould Hopper based Bernoulli polynomials of the second kind	$\frac{t}{\ln(1+t)}C_0(-x(\ln(1+t))^m)\exp(y\ln(1+t)+z(\ln(1+t))^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}b_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(e^{-\partial_y})+\frac{\partial_y}{e^{\partial_y}-1}-\frac{1}{e^{\partial_y}}$, $\hat{P}=e^{\partial_y}-1$
VIII.	${}_LH^{(m,r)}r_n(x,y,z):=$ Laguerre-Gould Hopper based Related polynomials	$\frac{2}{2+t}C_0(-x(\ln(1+t))^m)\exp(y\ln(1+t)+z(\ln(1+t))^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}r_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(e^{-\partial_y})-(e^{\partial_y}+1)^{-1}$, $\hat{P}=e^{\partial_y}-1$
IX.	${}_LH^{(m,r)}R_n(x,y,z):=$ Laguerre-Gould Hopper based Hahn polynomials	$\frac{1}{\sqrt{1+t^2}}C_0(-x(\arctan(t))^m)\exp(y\arctan(t)+z(\arctan(t))^r)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}R_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $(\cos^2(\partial_y))-\cos(\partial_y)\sin(\partial_y)$, $\hat{P}=\tan(\partial_y)$
X.	${}_LH^{(m,r)}R_n(a,x,y,z):=$ Laguerre-Gould Hopper based Shively's pseudo-Laguerre polynomials	$(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}C_0\left(-x\left(\frac{-4t}{(1+\sqrt{1-4t})^2}\right)^m\right)\times$ $\exp\left(\frac{-4yt}{(1+\sqrt{1-4t})^2}+z\left(\frac{-4t}{(1+\sqrt{1-4t})^2}\right)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}R_n(a,x,y,z)t^n$	$\hat{M}=-\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)$ $\left(\frac{(1-\partial_y)^3}{(1+\partial_y)}\right)+\left(\frac{1-\partial_y}{1+\partial_y}\right)^2$ $(2+(a-1)(1+\partial_y))$, $\hat{P}=\frac{1}{4}-\frac{1}{4}\left(\frac{1+\partial_y}{1-\partial_y}\right)^2$

Similarly, by taking $g(t)$ (or $A(t)$) and $f(t)$ (or $H(t)$) of the special polynomials belonging to the associated Sheffer family (Table 13 (I to IV)) in appropriate equations, we get the corresponding results for the members belonging to the LGHASP family.

We present these results along with the name and notation of the resultant special polynomial belonging to the LGHASP family in in Table 15.

Table 15. Certain results for the members belonging to the LGHASP family

S. No.	Name/Notation of the Resultant Special Polynomial	Generating Function	Multiplicative and Derivative Operators
I.	${}_LH^{(m,r)}M_n(x,y,z):=$ Laguerre-Gould Hopper based Mittag-Leffler polynomials	$C_0\left(-x\left(\ln\left(\frac{1+t}{1-t}\right)\right)^m\right)\exp\left(y\ln\left(\frac{1+t}{1-t}\right)+z\left(\ln\left(\frac{1+t}{1-t}\right)\right)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}M_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)\left(\frac{\left(1+e^{\partial_y}\right)^2}{2e^{\partial_y}}\right)$, $\hat{P}=\frac{e^{\partial_y}-1}{e^{\partial_y}+1}$
II.	${}_LH^{(m,r)}\varphi_n(x,y,z):=$ Laguerre-Gould Hopper based Exponential polynomials	$C_0\left(-x(e^t-1)^m\right)\exp\left(y\left(e^t-1\right)+z(e^t-1)^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}\varphi_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)(1+\partial_y)$, $\hat{P}=\ln(1+\partial_y)$
III.	${}_LH^{(m,r)}(x,y,z)_n:=$ Laguerre-Gould Hopper based Lower factorial polynomials	$C_0\left(-x(\ln(1+t))^m\right)\exp\left(y\ln(1+t)+z(\ln(1+t))^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}(x,y,z)_n\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)(e^{-\partial_y})$, $\hat{P}=e^{\partial_y}-1$
IV.	${}_LH^{(m,r)}p_n(x,y,z):=$ Laguerre-Gould Hopper based Bessel polynomials	$C_0\left(-x(1-\sqrt{1-2t})^m\right)\times$ $\exp\left(y\left(1-\sqrt{1-2t}\right)+z(1-\sqrt{1-2t})^r\right)$ $=\sum_{n=0}^{\infty}{}_LH^{(m,r)}p_n(x,y,z)\frac{t^n}{n!}$	$\hat{M}=\left(y+mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+rz\frac{\partial^{r-1}}{\partial y^{r-1}}\right)\frac{1}{1-\partial_y}$, $\hat{P}=-\frac{1}{2}\partial_y^2+\partial_y$

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